Three-variable statements of set-pairing

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and to MURST / MIUR 40% : *Aggregate- and number-reasoning* ⋯

☯ Pairing in antiquity ☯

In the first place, there were three kinds of human beings, not merely the two sexes, male and female, as at present: there was a third kind as well, which had equal shares of the other two, [...]*. Secondly, the form of each person was round all over, with back and sides encompassing it every way,* [...]*. Terrible was their might and strength, and the thoughts of their hearts were great, that they even conspired against the gods.*

Plato, *Symposium*

Toolkit for weak aggregate theories

(E) $\forall x \forall y \left(\forall v\, (v \in x \leftrightarrow v \in y) \rightarrow x = y\right)$

(N) $\exists z \forall v \neg v \in z$

(P) $\forall x \forall y \exists p \forall v \left(v \in p \leftrightarrow (v = x \vee v = y)\right)$

(W) $\forall x \forall y \exists w \forall v \left(v \in w \leftrightarrow (v \in x \vee v = y)\right)$

(L) $\forall x \forall y \exists \ell \forall v \left(v \in \ell \leftrightarrow (v \in x \,\&\, \neg v = y)\right)$

(R) $\forall x \exists r \left((r \in x \vee r = x) \,\&\, \neg \exists v\, (v \in r \,\&\, v \in x)\right)$

($\mathbf{A}^n$) $\forall x_0 \cdots \forall x_n\, (x_0 \in x_1 \in \cdots \in x_n \rightarrow \neg x_n = x_0)$

$n = 0, 1, 2, \ldots$

Universes of aggregates

All of the above sentences are provable within

- full Zermelo-Fraenkel set theory ZF

  (Note: $\frac{\textbf{(N)}}{\emptyset} = \frac{\textbf{(P)}}{\{x,y\}} = \frac{\textbf{(W)}}{x \cup \{y\}} = \frac{\textbf{(L)}}{x \setminus \{y\}}$)

- Tarski's theory of hereditarily finite sets

  (equipollent to Peano arithmetic)

By leaving some of these sentences *out* of our selection of axioms, we can frame our investigation inside less classical, but nevertheless useful, variants of set theory. E.g.,

- multisets do not meet *extensionality*, **(E)**;

- hypersets do not meet *regularity*, **(R)**, or even *acyclicity*, **(A)**

Through Skolemization of (N), (W), (L) . . .

. . . one obtains a constant, $\emptyset$, and dyadic operation symbols, with and less, designating the null set and single-element addition and removal

$$a, b \xmapsto{\text{with}} a \cup \{\, b \,\} \quad \text{and} \quad a, b \xmapsto{\text{less}} a \setminus \{\, b \,\},$$

respectively. We can continue with:

$$
\begin{aligned}
\{x, y\} &=: (\emptyset \ \text{with} \ x) \ \text{with} \ y & (1)\\
(x, y) &=: \Big\{\{x, y\}, \{x, x\}\Big\} & (2)\\
x \yinyang y &=: \{x \ \text{less} \ y, \ x \ \text{with} \ y\} & (3)\\
\langle x, y \rangle &=: \{y, y\} \yinyang x & (4)\\
[x, y] &=: (x \yinyang y) \yinyang x & (5)\\
\lfloor x, y \rfloor &=: x \ \text{with} \ \Big(y \ \text{with} \ (y \ \text{with} \ x)\Big) & (6)\\
\lceil x, y \rceil &=: \Bigg\{\Big\{x, x\Big\}, \Big\{\Big\{x, x\Big\}, \Big\{y, \{y, y\}\Big\}\Big\}\Bigg\} & (7)
\end{aligned}
$$

Of these, only (2), (4), (5), (6), and (7) can be regarded as acceptable pairing operations in a *full-fledged* set theory:

From $\{x, y\}$, one cannot retrieve unambiguously x or y
(because $\{x, y\} = \{y, x\}$);

from $x☯y$, only y can be retrieved with certainty

Any theory of aggregates endowed with an acceptable pairing notion enables one to restate the theory in purely *equational, quantifier-free* terms

This *bridge* between two logical systems enables *experimental comparison* based on state-of-art automated proof-assistants

Quick historical survey

1. **(N)** & **(P)** is a modern recasting of the *axiom of elementary sets*, which came second (after extensionality) in Zermelo's theory (1908)

2. The first reduction of the *ordered pair* notion to unordered pair enters into set theory with Norbert Wiener, who seeks (1914) to bring Ernst Schröder's algebraic formalism (which we call map arithmetic) closer to a theory of classes

3. Kazimierz Kuratowski refines (∼1921) the ordered pair notion into

$$(\, x, \; y \,) \quad =: \quad \Big\{ \, \{ \, x, y \, \}, \{ \, x \, \} \Big\}$$

4. Alfred Tarski (in the 1940s)

- notices that the components x, y can be retrieved not only from $(\ x,\ y\ )$ but also from $(\ x,\ y\ ) \cup \{\emptyset\}$

- finds a roundabout way of stating

  (OP) $\quad \forall x \, \forall y \, \exists p \Big(p \setminus \{\emptyset\} = (\ x,\ y\ ) \Big)$

  in three variables (without derived constructs)

- checks that **(OP)** $\dashv\vdash$ **(P)**

Thereby, he succeeds in translating

the whole of ZF into map arithmetic

Arithmetic of maps: ‘logical’ axioms

$P \cup Q = Q \cup P$

$\overline{\overline{P} \cup Q} \cup \overline{\overline{P} \cup \overline{Q}} = P$

$(P \cup Q) \circ R = P \circ R \cup Q \circ R$

$P^{\smile\smile} = P$

$(P \circ Q)^{\smile} = Q^{\smile} \circ P^{\smile}$

$P \cup Q \cup R = P \cup (Q \cup R)$

$P \circ Q \circ R = P \circ (Q \circ R)$

$P \circ \iota = P$

$(P \cup Q)^{\smile} = P^{\smile} \cup Q^{\smile}$

$P^{\smile} \circ \overline{P \circ Q} \cup \overline{Q} = \overline{Q}$

Proper axioms of a weak aggregate theory

(E) $\overline{\in^{\smile} \circ \overline{\in} \cup \overline{\in}^{\smile} \circ \in} \cup \iota = \iota$

(A^n) $\underbrace{\in \circ \cdots \circ \in}_{n+1 \text{ factors}} \cup \overline{\iota} = \overline{\iota}$

etc.

Tarski's restatement of (P) in 3 var's is:

$$\textbf{(OP)} \qquad \forall x \, \forall y \, \exists q \left(q \, \pi_0 \, x \,\&\, q \, \pi_1 \, y \right)$$

where

$$q \, \pi_0 \, x \;\; \leftrightarrow: \;\; \begin{cases} \qquad \exists y \Big(x \in y \;\;\&\;\; y \in q \;\;\& \\ \qquad \neg \exists q \left(q \in y \;\;\&\;\; q \neq x \right) \Big) \;\;\& \\ \neg \exists y \Big(\exists x \left(y \in x \;\;\&\;\; x \in q \;\;\& \right. \\ \qquad \left. \neg \exists q \left(q \in x \;\;\&\;\; q \neq y \right) \right) \;\;\&\;\; y \neq x \Big) \end{cases}$$

and

$$q \, \pi_1 \, y \;\; \leftrightarrow: \;\; \begin{cases} \exists x \left(y \in x \;\;\&\;\; x \in q \right) \;\;\& \\ \neg \exists x \Big(\exists y \left(x \in y \;\;\&\;\; y \in q \right) \;\;\& \\ \neg \, q \, \pi_0 \, x \;\;\&\;\; x \neq y \Big) \end{cases}$$

Maddux' translation technique - I

Predicates π_0, π_1 which—like the derived predicates above—meet the abstract properties **(OP)**,

$$\forall q \forall x_1 \forall x_2 \left(q \, \pi_0 \, x_1 \;\&\; q \, \pi_0 \, x_1 \;\rightarrow\; x_1 = x_2 \right),$$

and

$$\forall q \forall y_1 \forall y_2 \left(q \, \pi_1 \, y_1 \;\&\; q \, \pi_1 \, y_2 \;\rightarrow\; y_1 = y_2 \right),$$

are called *conjugated (quasi-) projections* and are the key for translating each sentence of a first-order theory into an equivalent 3-variable sentence

Maddux' translation technique - II

Assume $L^{\smile} \circ L \cup R^{\smile} \circ R \subseteq \iota$, $\quad L^{\smile} \circ R = L \circ 1\!\!1 = R \circ 1\!\!1 = 1\!\!1$.
Let $i, j = 0, 1, 2, \ldots$

$$\mathsf{th}(L, R \,\|\, 0) \;=:\; L \qquad \mathsf{th}(L, R \,\|\, i+1) \;=:\; \mathsf{th}(R, R, i) \circ L$$

$$\mathsf{th2}(L, R, P \,\|\, i, j) \;=:\; \big(\mathsf{th}(L, R, i) \circ \mathsf{th}^{\smile}(L, R, j)\big) \cap P$$

$$\mathsf{sibs}(L, R \,\|\, [\,]) \;=:\; 1\!\!1$$

$$\mathsf{sibs}(L, R \,\|\, [\mathsf{v}_i | \overrightarrow{V}\,]) \;=:\; \mathsf{th2}\big(L, R, \mathsf{sibs}(L, R, \overrightarrow{V}\,), i, i\big)$$

$$\mathsf{mXpr}\big(L, R \,\|\, \mathsf{v}_i = \mathsf{v}_j\big) \;=:\; \mathsf{th2}(L, R, \iota, i, j) \circ 1\!\!1$$

$$\mathsf{mXpr}\big(L, R \,\|\, \mathsf{v}_i \in \mathsf{v}_j\big) \;=:\; \big((\mathsf{th}(L, R, i) \circ \in) \cap \mathsf{th}(L, R, j)\big) \circ 1\!\!1$$

$$\mathsf{mXpr}(L, R \,\|\, \neg\, \varphi) \;=:\; \overline{\mathsf{mXpr}(L, R, \varphi)}$$

$$\mathsf{mXpr}(L, R \,\|\, \varphi \;\&\; \psi) \;=:\; \mathsf{mXpr}(L, R, \varphi) \cap \mathsf{mXpr}(L, R, \psi)$$

$$\mathsf{mXpr}(L, R \,\|\, \exists\, \overrightarrow{V}\; \varphi) \;=:\; \mathsf{sibs}\big(L, R, \mathsf{freeVars}(\exists\, \overrightarrow{V}\; \varphi)\big) \circ \mathsf{mXpr}(L, R, \varphi)$$

$$\mathsf{Maddux}(L, R \,\|\, \chi) \;\leftrightarrow:\; \mathsf{mXpr}(L, R, \chi) = 1\!\!1$$

Can we find any simpler 3-variable rendering of set pairing ?

We can e.g. strengthen
(P) into **(N)** & **(W)** & **(L)**, and *take advantage of* **(E)**

It can in fact be shown that

$$\textbf{(N)} \,\&\, \textbf{(W)} \,\&\, \textbf{(L)} \;\dashv\vdash_{\textbf{(E)}}\; \overbrace{\forall x \,\forall y \,\exists d \; x \text{☯} y = d}^{\textbf{(D)}}$$

In primitive symbols, this rendering
(where v, w, ℓ can be renamed x, y, y) is

$$\textbf{(D)} \quad \forall x \,\forall y \,\exists d \Big(x \in d \,\&\, \forall v \Big(v = y \leftrightarrow \exists w (w \in d \,\&\, v \in w) \,\&\, \exists \ell (\ell \in d \,\&\, \neg v \in \ell) \Big) \Big)$$

Set pairing under (N) & (W) & (L) and (E)

To translate into equational form—via Roger Maddux' technique—any axiomatic theory extending **(N)** & **(W)** & **(L)** & **(E)**, it will now suffice to single out predicates λ, ρ which can be proved to be conjugated projections via the restrained inferential apparatus named $\mathcal{L}_3$ in Tarski&Givant87, under assumptions **(E)** & **(D)**

Here are the desired λ, ρ
(corresponding to the pairing function $[x, y]$):

$$
\begin{array}{lll}
v\,\mu\,d & \leftrightarrow: & \exists\, w\,(w \in d \;\&\; v \in w) \\
 & & \;\&\; \exists\, \ell\,(\ell \in d \;\&\; \neg v \in \ell) \\
d\,\lambda\,x & \leftrightarrow: & \forall\, v\Big(v = x \;\leftrightarrow\; v\,\mu\,d\Big) \\
q\,\rho\,y & \leftrightarrow: & \exists\, d\Big(d \in q \;\&\; d\,\lambda\,y \\
 & & \qquad \;\&\; \exists\, x\Big(x \in d \;\&\; \forall\, v\big(v\,\mu\,q \;\rightarrow\; v = x\,\big)\Big)\Big)
\end{array}
$$

Instead of directly deriving QProj(λ, ρ) from **(E)** & **(D)** in $\mathcal{L}_3$, the authors

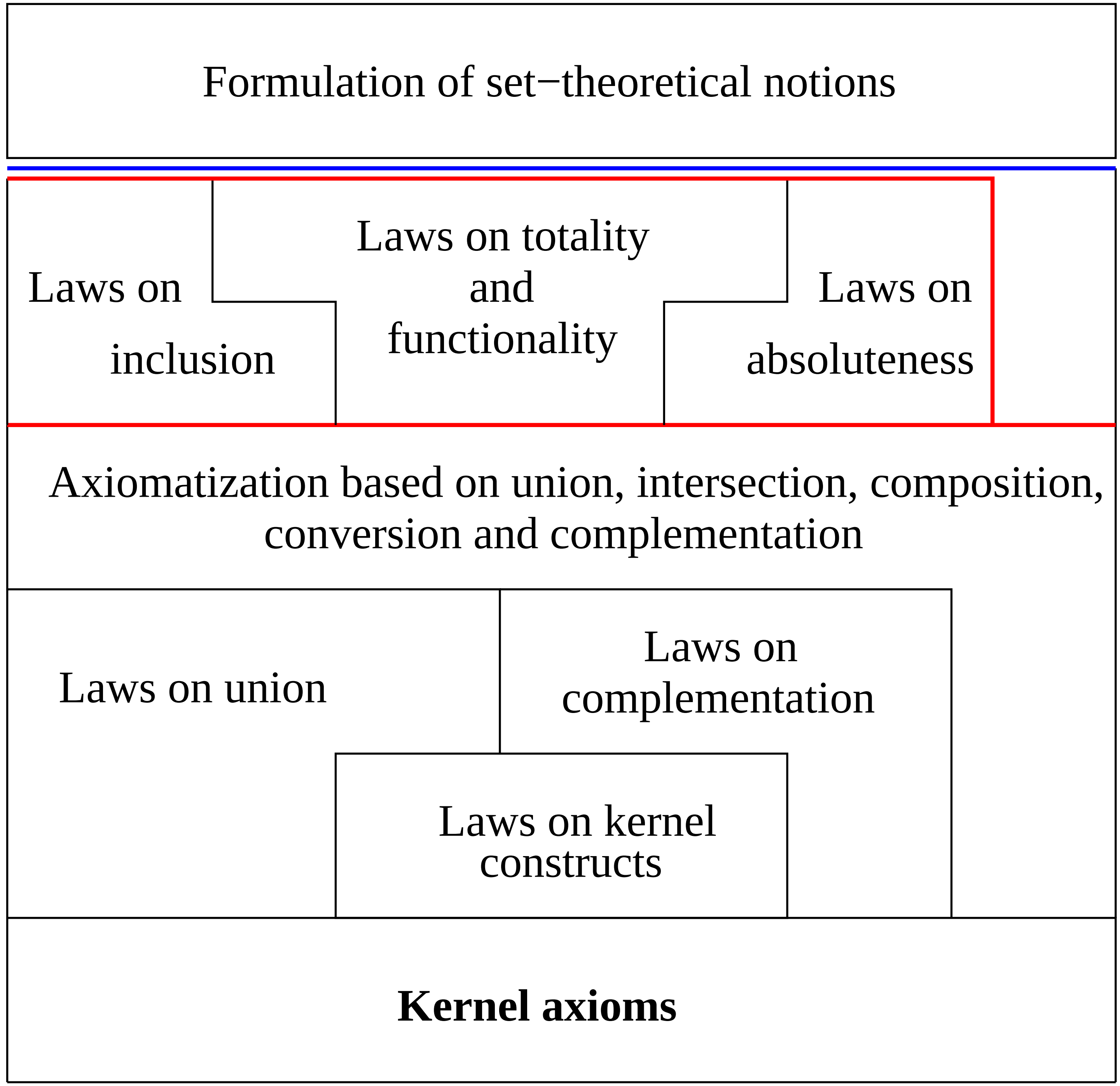
- translated λ, ρ, **(E)** & **(D)**, QProj(λ, ρ) into map arithmetic, e.g.,

$$\begin{array}{rcl} \mu & =: & \in \circ \in \cap \notin \circ \in \\ \lambda^{\smile} & =: & \mu - \bar{\boldsymbol{\iota}} \circ \mu \\ \rho & =: & (\in^{\smile} \cap \overline{\mu^{\smile} \circ \bar{\boldsymbol{\iota}}} \circ \in) \circ \lambda \\ \mathsf{QProj}(L, R) & \leftrightarrow: & L^{\smile} \circ L \cup R^{\smile} \circ R \subseteq \boldsymbol{\iota} \;\&\; L^{\smile} \circ R = \mathbb{1} \end{array}$$

- and then exploited a standard theorem-prover, Otter (from the Argonne National Laboratory)

This is a *prelude* to a wider experimentation related to equational formulations of set theories

Note: **(E)** & **(W)** & **(L)** *cannot* be stated in 3 var's


Formulation of set−theoretical notions
Laws on inclusion
Laws on totality and functionality
Laws on absoluteness
Axiomatization based on union, intersection, composition, conversion and complementation
Laws on union
Laws on complementation
Laws on kernel constructs
Kernel axioms

Set pairing under (A^5) & (W) & (L) & (E)

Here are conjugated projections α, β which correspond to the pairing function $\lfloor x, y \rfloor$:

$$\alpha \quad =: \quad \mathsf{syq}(\in \ \cap \ \in \circ \in \circ \in \circ \in, \in)$$
$$\beta \quad =: \quad \gamma_3 \circ \mathsf{syq}(\in \ \cap \ \in \circ \in, \in)$$

where

$$\mathsf{syq}(P, Q) \quad =: \quad \overline{P^{\smile} \circ \overline{Q}} \cap \overline{\overline{P}^{\smile} \circ Q}$$
$$\gamma_n \quad =: \quad \in^{\smile} - \Big(\in^{\smile} \circ (\bar{\iota} - \underbrace{\in \circ \cdots \circ \in}_{n \text{ factors}}) \Big)$$

The single-valuedness of α and β is easily derived (with Otter) from a 3-var statement of **(E)** & **(A^5)**

Then we must add

$$\textbf{(OP}_1\textbf{)} \qquad \alpha^{\smile} \circ \beta = \mathbb{1}$$

as an explicit axiom

Thanks to $\mathsf{QProj}(\alpha, \beta)$, getting a 3-variable translation of **(W)** & **(L)** becomes a routine matter

Set pairing under (R) & (N) & (W) & (L)

Here are conjugated projections $\mathsf{car}, \mathsf{cdr}$ which correspond to the pairing function $\lceil x, y \rceil$:

$$
\begin{array}{rcl}
\mathsf{arb} & =: & \mathsf{funcPart}(\in^{\smile} - \in^{\smile} \circ \in) \\
\mathsf{car} & =: & \mathsf{arb} \circ \mathsf{arb} \\
\mathsf{arb\_lessArb} & =: & \mathsf{syq}(\in - \mathsf{arb}^{\smile}, \in) \circ \mathsf{arb} \\
\mathsf{cdr} & =: & \mathsf{syq}(\in \circ \mathsf{arb\_lessArb}^{\smile} - \mathsf{arb}^{\smile}, \in) \circ \mathsf{car}
\end{array}
$$

where syq is as before and

$$\mathsf{funcPart}(P) =: P - P \circ \overline{\iota}$$

Since **(R)** is in three variables, and the single-valuedness of arb and car can be proved quite easily, we can handle this case like the preceding one

Conclusions

Experimentation with a typical theorem-prover indicates that equational formulations of aggregate theories, based on map arithmetic, can favorably compete with more conventional first-order formulations

(Similar indications come from the work of Johan G. F. Belinfante, carried out within the framework of Gödel-Bernays' class theory)

To achieve results in map arithmetic human guidance consists, instead of in pointing out key intermediate lemmas, in developing a systematic layered architecture of generic laws